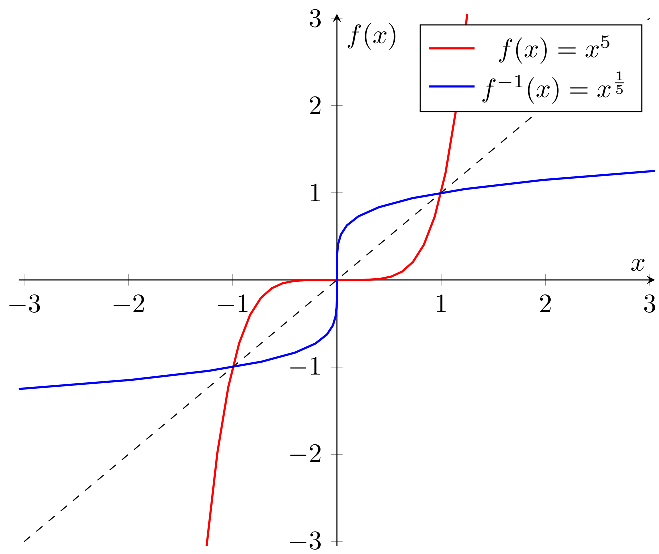
1 WS4 7.1 (A)

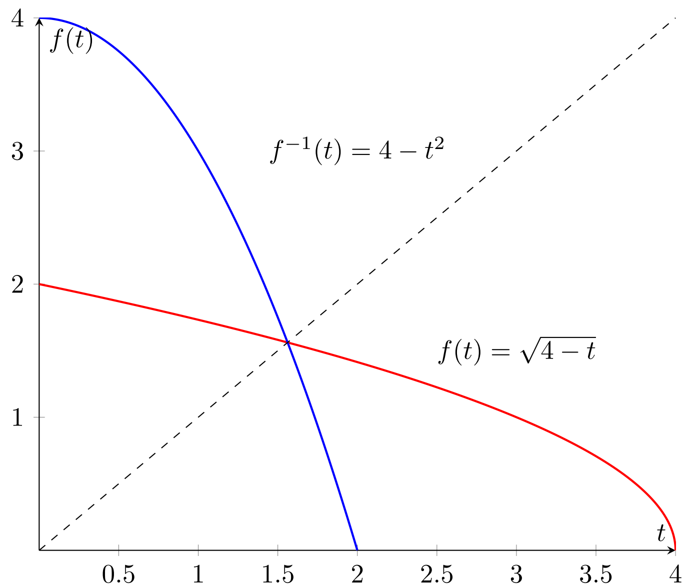
Do the following functions have an inverse? If one exists, give the domain and range:
1) $f(x) = x^5$
2) $f(t) = \sqrt{4-t}$
3) $f(x) = x + |x|$

1) $f(x) = x^5$. Note $f'(x) = 5x^4 > 0$ is positive for all nonzero x and is 0 only at 0. Then, by Theorem 4.7 (p227) in our book, if $f'(x) > 0$ on $I = (a, b)$ except for a finite number of points x in $I = (a, b)$, then f is increasing on I.

Then, by Theorem 7.3 (p435) every increasing function has an inverse. Therefore f has an inverse on its domain (we could also have applied the horizontal line test, the graph of x^5 looks like the graph of x^3, x^7, x^9 etc. even though it looks constant near 0). Since the domain and range of $f(x) = x^5$ are $(-\infty, \infty)$, the domain and range of $f^{-1}(x) = x^{\frac{1}{5}}$ are $(-\infty, \infty)$ as well, because Domain(f) = Range(f^{-1}) and vice versa.



2) $f(t) = \sqrt{4-t}$. Note the domain and range of $f(t)$ are $t \le 4$ and $[0, \infty)$ respectively, and f satisfies the Horizontal Line Test (HLT) on its domain.
To see this, note $f(t)$ is a transformation of the graph of $\sqrt{t}$, first we could translate right by 4: $t \to t - 4$ then reflect it across the y-axis: $\sqrt{t} \to \sqrt{t-4} \to \sqrt{-(t-4)} = f(t)$ or we could first do a reflection across y then translate left by 4: $\sqrt{t} \to \sqrt{-t} \to \sqrt{-(t-4)} = f(t)$. Either way we obtain a transformation of $f(t) = \sqrt{t}$ whose graph satisfies HLT:



To find f^{-1}, we switch x and y in the equation $f(x) = y \to f(y) = x$ and solve for y:

$$f(y) = x = \sqrt{4-y}$$
$$\implies x^2 = 4 - y$$
$$\implies y = 4 - x^2 = f^{-1}(x).$$

Since the domain and range of $f(t) = \sqrt{4-t}$ are $(-\infty, 4]$ and $[0, \infty)$ respectively, the domain and range of f^{-1} are $[0, \infty)$ and $(-\infty, 4]$ respectively.

3) $f(x) = x + |x|$. First note $x + |x| = 0$ if $x \le 0$ while $x + |x| = 2x$ if $x > 0$ since $|x| = x$ there. Thus $f(x)$ does not have an inverse on its full domain since it is constant for all negative x (different values of x are sent to the same value).

It is invertible on the set $[0, \infty)$ however, with inverse $g(x) = \frac{x}{2}$ for $0 \le x$ (notice the slope of g is the inverse of the slope of f).

2 WS4 7.1 (B)

Calculate $(f^{-1})'(b)$:
4) $f(x) = x^3 + 7$, $b = 6$
5) $f(x) = \tan(x)$ on $-\frac{\pi}{2} < x < \frac{\pi}{2}$ for $b = \sqrt{3}$
6) $f(x) = 4\ln(x)$ for $b = 0$.

4) $f(x) = x^3 + 7$, $b = 6$
Well, first we solve for a in $6 = b = f(a)$ to find the point (a, b) on the function in order to use the formula $(f^{-1})'(b) = \frac{1}{f'(a)}$.

Note $f(a) = a^3 + 7 = 6 \implies a^3 = -1 \implies a = -1$. Now, we calculate $f'(x) = 3x^2$, telling us $f'(a) = f'(-1) = 3(-1)^2 = 3$. Thus $(f^{-1})'(b) = (f^{-1})'(6) = \frac{1}{f'(a)} = \frac{1}{3}$.

5) $f(x) = \tan(x)$ on $-\frac{\pi}{2} < x < \frac{\pi}{2}$ for $b = \sqrt{3}$
Again, first we solve for a in $6 = b = f(a)$ to find the point (a, b) on the function in order to use the formula

$(f^{-1})'(b) = \frac{1}{f'(a)}$.
Note $f(a) = \tan(a) = \sqrt{3} = \frac{\sqrt{3}}{1} = \frac{\text{hyp}_a}{\text{adj}_a}$ for $-\pi/2 < a < \pi/2$ telling us that $a = \frac{\pi}{3}$.
We'll also need $f'(x) = \sec^2(x)$, so $f'(a) = f'(\pi/3) = \sec^2(\pi/3) = (\frac{\text{hyp}_{\pi/3}}{\text{adj}_{\pi/3}})^2 = (\frac{2}{1})^2 = 4$ (remember $\pi/3 = 30°$ for a 30-60-90 triangle).

Now, we apply the formula $(f^{-1})'(b) = (f^{-1})'(\sqrt{3}) = \frac{1}{f'(a)} = \frac{1}{f'(\pi/3)} = \frac{1}{4}$.

6) $f(x) = 4\ln(x)$ for $b = 0$.

Again, first we solve for a in $6 = b = f(a)$ to find the point (a, b) on the function in order to use the formula $(f^{-1})'(b) = \frac{1}{f'(a)}$.

Note $b = 0 = f(a) = 4\ln(a) = 0 \iff \ln(a) = 0 \iff a = 1$ (because for any $c > 0$, $\log_c(1) = 0 \iff c^0 = 1$). Now, since, $f'(x) = \frac{4}{x}$, we have $f'(a) = f'(1) = \frac{4}{1} = 4$. Then, $(f^{-1})'(b) = \frac{1}{f'(a)} = (f^{-1})'(0) = \frac{1}{4}$.

3 WS4 7.1 (C)

Can a polynomial of even degree have an inverse? explain.

A polynomial of even degree cannot have an inverse on its entire domain because if $p(x)$ is a polynomial of even degree, $\lim_{x\to\infty} p(x) = \infty$ and $\lim_{x\to-\infty} p(x) = \infty$. Thus, the horizontal line test fails for alll "big enough" lines $y = a$ where $a > 0$, by the intermediate value theorem (IVT): a continuous function on $[a, b]$ assumes all values c between $f(a)$ and $f(b)$.

Split $(-\infty, \infty)$ up into $(-\infty, 0]$ and $[0, \infty)$. Then since $\lim_{x\to-\infty} p(x) = \infty$, apply IVT to the interval $(-\infty, 0]$ so all values between $p(0)$ and ∞ are obtained by $p(x)$ with $x < 0$.

Similarly, applying the IVT to $[0, \infty)$, all values between $p(0)$ and ∞ are obtained by $p(x)$ with positive x. Then, if $C \geq p(0)$, the line $y = C$ must intersect the graph of $p(x)$ twice, once at a negative x value ($x = -B$ for some $B > 0$) and once at a positive x value ($x = A$).

4 WS4 7.1 (D)

Suppose f has a continuous derivative on the interval $[0, 6]$. Assume also that f' is increasing on $[0, 4]$, f' is decreasing on $[4, 6]$, and $f'(0) = -1$, $f'(3) = 0$, $f'(4) = 2$, and $f'(5) = 0$. On what subintervals does f have an inverse?

Note that f' increasing on $[0, 4]$ but $f'(0) = -1$, $f'(3) = 0$ means that $f'(x) < 0$ on [0,3] except for at a finite number of points (at $x = 3$) since the problem states that f' is continuous. Thus f is decreasing on [0,3] by Theorem 4.7 (p227) in our book, so f has an inverse here by Theorem 7.3 (p435) in our book.

Then, on [3, 4], f' is still increasing (it increases on [0,4]), so $f'(x) > 0$ on $[3, 4]$ except at $x = 3$. Further, $f'(x) > 0$ on [4,5] except at $x = 5$, since it is decreasing but still positive until $f'(5) = 0$. Thus $f'(x) > 0$ on $[3, 5]$ except for at the finite number of points $x = 3$ and $x = 5$, so f is increasing on [3,5], hence has an inverse on this subinterval.

Finally, on $[5, 6]$, $f'(x) < 0$ except for the point $x = 5$ ($f'(5) = 0$), since f' decreases on $[4, 6]$. Thus f decreases on [4,6], hence has an inverse here.

Then, the subintervals where f has an inverse are [0,4], [4,5], and [5,6].

5 WS4 7.2 (A)

1) Show that the maximum value of the normal density function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$ is $f(\mu) = \frac{1}{\sigma\sqrt{2\pi}}$.

Well, first we find f' to use the first derivative test to find a maximum. We'll use the formula

$$\tfrac{d}{dx} e^{f(x)} = f'(x) \cdot e^{f(x)}$$

treating σ and μ as constants. Then

$$f'(x) = \tfrac{-2(x-\mu)}{2\sigma^2} \cdot \tfrac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} = \tfrac{-(x-\mu)}{\sigma^3\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}.$$

Note $f'(\mu) = 0 = \frac{-(\mu-\mu)}{\sigma^3\sqrt{2\pi}} e^{\frac{-(\mu-\mu)^2}{2\sigma^2}}$ is the only 0 of f'. This indicates a local extrema.

Left of $x = \mu$, the slope is positive since $e^x > 0$ for all x, while $\sigma > 0$ (σ is a standard deviation), so with the trial point $\mu - 1$ we have $-((\mu - 1) - \mu) = 1 > 0$ tells us f' is positive on the interval $(-\infty, \mu)$.

Similarly, on the interval to the right of μ, $f'(\mu + 1) < 0$. Thus $f(\mu)$ is the maximum of f by the first derivative test.

6 WS4 7.2 (B)

Find the area A of the region bounded by the graphs of $y = 3e^x$ and $y = 2 + e^{2x}$.

We solve for the intersection points of the two functions: $3e^x = 2 + e^{2x} \implies e^{2x} - 3e^x + 2 = 0$. Let $X = e^x$ so that $X^2 - 3X + 2 = 0 = (X - 2)(X - 1)$, so $X = 1 = e^a \implies a = 0$, $X = 2 = e^b \implies b = \ln(2)$ are our bounds.

To see which function is on top over the interval $[0, \ln(2)]$, let's try a trial point to compare: $3e^1 = 3e$, while $2 + e^{2\cdot 1} = 2 + e^2$. This is a little difficult to decide which one is bigger.

Lets try $x = \ln\frac{3}{2}$ ($\ln(x)$ is increasing, so $0 < \ln\frac{3}{2} < \ln(2)$, and we're still in the interval). Then $3e^{\ln\frac{3}{2}} = 3\cdot\frac{3}{2} = \frac{9}{2} = 4+\frac{1}{2}$, while $2+e^{2\ln\frac{3}{2}} = 2+(e^{\ln\frac{3}{2}})^2 = 2+(\frac{3}{2})^2 = 2+\frac{9}{4} = \frac{8}{4}+\frac{9}{4} = \frac{17}{4} = 4+\frac{1}{4}$. Since $4+\frac{1}{2} > 4+\frac{1}{4}$, $3e^x$ is bigger than $2 + e^{2x}$ on $[0, \ln 2]$.

Now, to find the area A of the region, we integrate top minus bottom:

$$A = \int_{x=0}^{x=\ln 2} [(3e^x) - (2 + e^{2x})]dx = [3e^x - 2x - \frac{1}{2}e^{2x}]_0^{\ln 2} = [3e^{\ln 2} - 2\ln 2 - \frac{1}{2}e^{2\ln 2}] - [3e^0 - 2\cdot 0 - \frac{1}{2}e^{2\cdot 0}] \tag{1}$$

$$= [3\cdot 2 - 2\ln 2 - \frac{1}{2}2^2] - [1 - 0 - \frac{1}{2}] \tag{2}$$

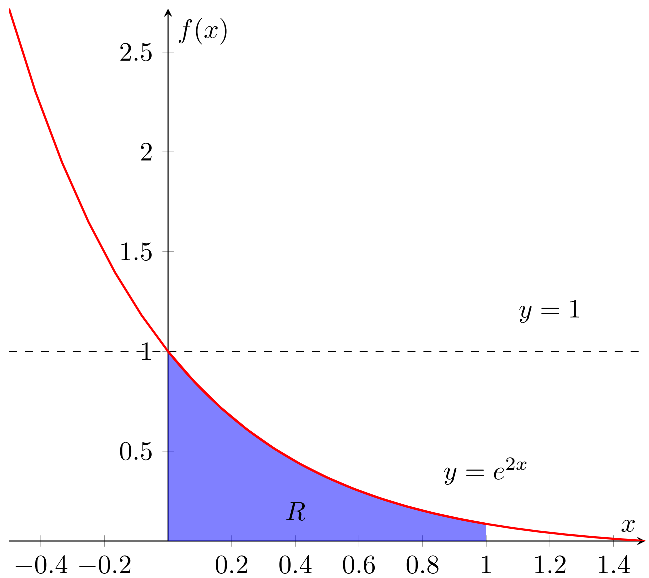
$$= [4 - 2\ln 2] - [\frac{1}{2}] \tag{3}$$

$$= \frac{7}{2} - 2\ln 2 \tag{4}$$

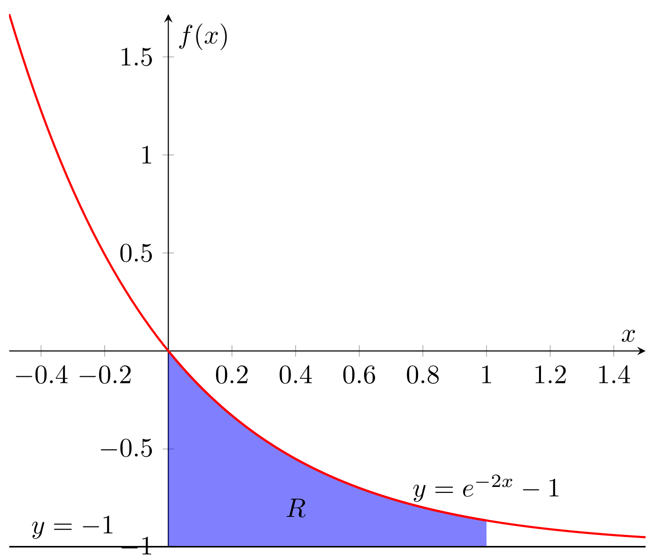
7 WS4 7.2 (C)

Find the volume V of the solid obtained by revolving about the line $y = 1$ the region between the graph of the equation $y = e^{-2x}$ and the x-axis on the interval [0,1].

First, I'll plot the region R that we'll rotate:



Note that rotating R around the line $y = 1$ is the same as rotating the area between the functions $f(x) = e^{-2x} - 1$ and $g(x) = -1$ about the line $y = 0$ so we can use washers about the x-axis (i.e. shifting all y-values down by 1):



We'll obtain V by the Washer method applied to OUTER function $g(x) = -1$ and INNER function

$f(x) = e^{-2x} - 1$ (make sure you understand this) on the interval [0,1]:

$$V = \pi \int_{x=0}^{x=1} [(outer)^2 - (inner)^2]dx = \pi \int_{x=0}^{x=1} [(-1)^2 - (e^{-2x} - 1)^2]dx \tag{5}$$

$$= \pi \int_{x=0}^{x=1} [1 - (e^{-4x} - 2e^{-2x} + 1)]dx \tag{6}$$

$$= \pi \int_{x=0}^{x=1} (-e^{-4x} + 2e^{-2x})dx \tag{7}$$

$$= \pi[-\frac{e^{-4x}}{-4} + 2\frac{e^{-2x}}{-2}]_0^1 \tag{8}$$

$$= \pi[(\frac{e^{-4}}{4} - e^{-2}) - (\frac{e^0}{4} - e^0)] \tag{9}$$

$$= \pi[\frac{3}{4} - e^{-2} + \frac{e^{-4}}{4}] = \frac{\pi(3 - 4e^{-2} + e^{-4})}{4} \tag{10}$$